§4.2 Correspondence Cartier Divisors – Line Bundles

Let $D = \sum_{V_i} a_i V_i$ be a Cartier Divisor of X

Then it there exists an open cover $\{U_\alpha\}_\alpha$ of X such that $V_i = (g_{i\alpha})$ on U_α. We can define

$$f_\alpha = \prod_i g_{i\alpha}^{a_i} \in \mathcal{M}^*(U_\alpha)$$

and f_α / f_β never vanishes on $U_\alpha \cap U_\beta$.

This means $\{(U_\alpha, f_\alpha)\}_\alpha \in Z^0(\underline{U}, \mathcal{M}^*/\mathcal{O}^*)$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$H^0(\underline{U}, \mathcal{M}^*/\mathcal{O}^*)$$

Conversely, given $\{(U_\alpha, f_\alpha)\}_\alpha \in H^0(\underline{U}, \mathcal{M}^*/\mathcal{O}^*)$, then $f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and so

$$\mathrm{ord}_V(f_\beta) = \mathrm{ord}_V(f_\alpha)$$

This means we can define

$$D := \sum_V \mathrm{ord}_V(f_\alpha) \cdot V$$

so $\boxed{\mathrm{Div}(X) = H^0(X, \mathcal{M}^*/\mathcal{O}^*)}$.

Instead, it is easy to see that

$$\boxed{\mathrm{PDiv}(X) = \frac{H^0(X, \mathcal{M}^*)}{H^0(X, \mathcal{O}^*)}}$$

Let D be a Cartier divisor of X, $D=(f_\alpha)_\alpha$, then we can define

$$f_{\alpha\beta} := f_\alpha / f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$$

We observe that

$$f_{\beta\gamma} \cdot f_{\alpha\gamma}^{-1} f_{\alpha\beta} = \frac{f_\beta}{f_\gamma} \cdot \frac{f_\gamma}{f_\alpha} \cdot \frac{f_\alpha}{f_\beta} = 1$$

So $\{f_{\alpha\beta}\}_{\alpha\beta}$ is a cocycle of $H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$

We have constructed a function

$$\begin{aligned} \mathrm{Div}(X) &\longrightarrow H^1(X, \mathcal{O}^*) \\ D=(f_\alpha) &\longmapsto \{f_{\alpha\beta} = \tfrac{f_\alpha}{f_\beta}\}_{\alpha\beta} \end{aligned}$$

Clearly, this function is an homomorphism of groups. What is its kernel?

Let $D=(f_\alpha)$ be such that $\{f_{\alpha\beta} = \frac{f_\alpha}{f_\beta}\}_{\alpha\beta}$ is trivial on $H^1(X, \mathcal{O}^*)$.

Then $f_{\alpha\beta} = (\delta h)_{\alpha\beta}$ for $h \in C^0(\underline{U}, \mathcal{O}^*)$

$$\Rightarrow \; f_\alpha / f_\beta = \frac{h_\beta}{h_\alpha} \; \Rightarrow \; f_\alpha h_\alpha = h_\beta f_\beta \text{ on } U_\alpha \cap U_\beta$$

Thus, we consider the global meromorphic function

$$f := \{(U_\alpha, f_\alpha h_\alpha)\}_\alpha \quad \text{on } X$$

and we observe that

$$\mathrm{div}(f) = \sum_V \underbrace{\mathrm{ord}_V(f)}_{\mathrm{ord}_V(f_\alpha h_\alpha) = \mathrm{ord}_V(f_\alpha)} V = D$$

We have obtained an exact sequence of groups:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{PDiv}(X) \arrow[r, hook] \arrow[d, phantom, "\parallel"] & \mathrm{Div}(X) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{Pic}(X) \quad (*) \\
 & \dfrac{H^0(X, \mathcal{M}^*)}{H^0(X, \mathcal{O}^*)} & H^0(X, \mathcal{M}^*/\mathcal{O}^*) &
\end{tikzcd}$$

We are interested to understand when the right map is surjective. To understand this, we consider the exact sequence of sheaves

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$$

which induces a long exact sequence in cohomology:

$$0 \to H^0(X, \mathcal{O}^*) \to H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{M}^*/\mathcal{O}^*) \xrightarrow{\delta^*} \mathrm{Pic}(X) \to H^1(X, \mathcal{M}^*) \to \cdots$$

and so the exact sequence

$$\frac{H^0(X,\mathcal{M}^*)}{H^0(X,\mathcal{O}^*)} \hookrightarrow H^0(X, \mathcal{M}^*/\mathcal{O}^*) \xrightarrow{\delta^*} \mathrm{Pic}(X) \to H^1(X,\mathcal{M}^*) \to \cdots$$

As we can expect, δ^* is exactly the map (*)
So that it is surjective $\Leftrightarrow$

$$H^1(X,\mathcal{O}^*) \longrightarrow H^1(X,\mathcal{M}^*)$$

is the trivial map.

Theorem (Chen-Kerr-Lewis, "The sheaf of nonvanishing meromorphic functions in the projective algebraic case is NOT acyclic")

If X is a smooth projective variety, then

$$H^1(X,\mathcal{O}^*) \longrightarrow H^1(X,\mathcal{M}^*)$$

is trivial, so

$$\mathrm{Pic}(X) \cong_{\delta^*} \frac{\mathrm{Div}(X)}{\mathrm{PDiv}(X)}$$

Furthermore, $H^1(X,\mathcal{M}^*) = 0 \Leftrightarrow \dim(X) = 1$
(so X is a Riemann Surface)

Warning! In the folklore, it was taken for granted that, for smooth projective varieties, the fact that every line bundle comes from a divisor of X was a consequence of $H^1(X,\mathcal{M}^*) = 0$.

This is FALSE unless X is a Riemann Surface (i.e. $\dim(X) = 1$).

Def The equivalence relation induced by $PDiv(X)$ on $Div(X)$ is called linear equivalence relation:

$$D \sim D' \overset{\text{def}}{\iff} D - D' = div(f) \text{ for some global merom. function } f.$$

We finally have proved the following isomorphisms of groups:

$$\left\{ \begin{array}{c} \text{invertible sheaves} \\ \text{on } X \end{array} \right\} \Big/ \text{iso} \qquad \mathcal{O}(D) := \Gamma(-, [D]) \text{ sheaf of (holom.) sections of } [D] \text{, namely}$$

$$\Gamma(U, [D]) = \left\{ \{(U \cap U_\alpha, s_\alpha)\}_\alpha \;\middle|\; \begin{array}{l} s_\beta = \frac{f_\beta}{f_\alpha} s_\alpha \\ s_\alpha \in \mathcal{O}(U \cap U_\alpha) \end{array} \right\}$$

$$\updownarrow \qquad\qquad\qquad \uparrow$$

$$\left\{ \begin{array}{c} \text{line} \\ \text{bundles on } X \end{array} \right\} \Big/ \text{iso} \qquad [D] = \bigcup_\alpha U_\alpha \times \mathbb{C} \Big/ \sim$$

where $(p, \lambda) \sim (q, \mu) \iff p = q \in U_\alpha \cap V_\beta$ and $\mu = \frac{f_\beta}{f_\alpha} \lambda$

$$\updownarrow \qquad\qquad\qquad \uparrow \quad \text{abuse of notation}$$

$$Pic(X) \qquad\qquad [D] = \{(U_\alpha \cap V_\beta, f_\alpha / f_\beta)\}$$

$$\updownarrow \qquad\qquad\qquad \uparrow$$

$$\frac{Div(X)}{PDiv(X)} \qquad\qquad D = \{(U_\alpha, f_\alpha)\}$$

Example We have already studied the tautological bundle of $\mathbb{P}^n$, which is a line bundle with cocycles $\left\{ g_{ji} = \frac{x_j}{x_i} \right\}_{ji}$ on $U_{x_j} \cap U_{x_i}$.

We are interested to understand which is the class divisor D associated to this line bundle.

Let us consider $H = (x_1 = 0) = \left(\mathcal{U}_{x_i}, \frac{x_1}{x_i}\right)_i$.

Then the cocycles of H are $g_{ji} = \frac{\frac{x_1}{x_j}}{\frac{x_1}{x_i}} = \frac{x_i}{x_j}$ on $\mathcal{U}_{x_i} \cap \mathcal{U}_{x_j}$.

Thus, the cocycles of $-H$ are $g_{ji} = \frac{x_j}{x_i}$ on $\mathcal{U}_{x_i} \cap \mathcal{U}_{x_j}$.

We have proved that the tautological bundle of $\mathbb{P}^n$ is $[-H]$.

Thus $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z} \cdot H$ where H is an hyperplane section.

Def On X we always have the canonical bundle ω_X. We say that $D \in \operatorname{Div}(X)$ is a canonical divisor if $[D] = \omega_X$.

The class of a canonical divisor in $\frac{\operatorname{Div}(X)}{\operatorname{PDiv}(X)}$ is denoted by K_X.

Example $X = \mathbb{P}^n$, ω_X has cocycles $\{\det J^{-1}(\varphi_j \circ \varphi_i^{-1})\}_{ji}$ on $\mathcal{U}_{x_j} \cap \mathcal{U}_{x_i}$.

$\det J^{-1}(\varphi_j \circ \varphi_i^{-1}) = (-1)^{i+j}\left(\frac{x_j}{x_i}\right)^{n+1}$

Thus, $\omega_{\mathbb{P}^n} = -(n+1)H$.

We need a constructive way to pass from a line bundle L to an associate divisor D with $[D]=L$.

Def Assume that L has cocycles $\{g_{\alpha\beta}\}_{\alpha\beta}$.
A local meromorphic section of L on U is a collection $\{(U\cap U_\alpha), s_\alpha)\}_\alpha$ where s_α is a meromorphic fuction on $U\cap U_\alpha$ and

$$s_\beta = g_{\beta\alpha}\, s_\alpha \quad \text{on } U\cap U_\alpha \cap U_\beta.$$

To any global meromorphic section s of L we can define the divisor

invariant by s_α.

$$\operatorname{div}(s) := \sum_V{}' \operatorname{ord}_V(s_\alpha)\cdot V$$

By construction, we have $[\operatorname{div}(s)] = L$.

Given a line bundle L, we can always construct a meromorphic section s for which $[\operatorname{div}(s)] = L$. Roughly speaking, if L has transition functions $\{g_{\beta\alpha}\}_{\beta\alpha}$ on $\underline{U}=\{U_\alpha\}_\alpha$, then fixed U_γ, we define a global meromorphic section $s := \{(U_\alpha, s_\alpha)\}_\alpha$ of L

where $s_\alpha(x) := g_{\alpha\gamma}(x)$ for $x\in U_\alpha$ (in there is some theoretical problem here, but this usually works) (why $g_{\alpha\gamma}$ is mero. on U_α?)

Then, $\forall x \in U_\alpha\cap U_\beta$ we have $s_\beta(x) = g_{\beta\gamma}(x) = g_{\beta\alpha}(x) g_{\alpha\gamma}(x) = g_{\beta\alpha}(x)\cdot s_\alpha(x)$
$\Rightarrow$ s is a global merom. form of L.

§4.3 Pullback of a divisor and Hurwitz formula

Let $X \xrightarrow{f} Y$ be a morphism. As already discussed, we have

$$\begin{array}{ccc}
\{\text{invertible sheaves of } Y\}/\text{iso} & \xrightarrow{f^*} & \{\text{invertible sheaves of } X\}/\text{iso} \\
\mathcal{F} & \longmapsto & f^*\mathcal{F} \\
\updownarrow & & \updownarrow \\
\{\text{line bundles on } Y\}/\text{iso} & \xrightarrow{f^*} & \{\text{line bundles on } X\}/\text{iso} \\
(E \to Y) & \longmapsto & f^*E \to X \\
\updownarrow & & \updownarrow \\
H^1(Y, \mathcal{O}^*) & \xrightarrow{f^*} & H^1(X, \mathcal{O}^*) \\
\{g_{\alpha\beta}\}_{\alpha\beta} & \longmapsto & \{g_{\alpha\beta} \circ f\}_{\alpha\beta} \\
\updownarrow & & \updownarrow \\
\mathrm{Div}(Y)/\mathrm{PDiv}(Y) & \xrightarrow{f^* ?} & \mathrm{Div}(X)/\mathrm{PDiv}(X)
\end{array}$$

(inverse image sheaf)

(pullback bundle)

Pullback divisor?

A natural definition of f^*D seems pretty easy:

$$f^*D = \{(f^{-1}(U_\alpha), f_\alpha \circ f\}_\alpha$$

However, this definition does not make sense if $\mathrm{Im}(f) \subseteq \mathrm{supp}(D)$ as $f_\alpha \circ f$ becomes identically zero on X.

Notice that if f is dominant then $\mathrm{Im}(f) \not\subseteq \mathrm{supp}(D)$ so we can always define f^*D.

Instead, we can avoid in general this problem defining f^*D as the class divisor associated to $f^*[D]$.

Clearly, in this case we lose information as we can only construct a linearly equivalent class in $\frac{Div(X)}{PDiv(X)}$ and not a divisor in $Div(X)$ such as before.

Given a dominant morphism $\pi: X \to Y$, a natural question is to determine a relationship between

$$K_X \quad \text{and} \quad \pi^* K_Y$$

Theorem (Hurwitz Formula)

Let $\pi: X \to Y$ be a dominant morphism of smooth projective varieties with the same dimension n.
Let V_J be a irreducible prime divisor of X whose image with respect to π is a irr. prime divisor W_J.

Let us define the ramification index of V_J as the coefficient $e_J \geq 0$ of V_J appearing on $\pi^* W_J$.

Let E_i be the prime divisors of X contracted by π. Then $K_X = \pi^* K_Y + \underbrace{\sum_J (e_J - 1) V_J + \sum_i r_i E_i}_{R}$ for some $r_i \geq 0$.

The effective divisor R is called Ramification divisor of π.

proof: Let us consider a global meromorphic section ω exhibiting ω_Y as $\omega_Y = [\mathrm{div}(\omega)]$.

We can construct an open cover $\{U_\alpha\}_\alpha$ of Y such that it there exists an open cover $\{V_\alpha\}_\alpha$ of X with $V_\alpha \subseteq \pi^{-1}(U_\alpha)$ for which

- $\pi_\alpha := \Psi_\alpha \circ \pi \circ \varphi_\alpha^{-1}$ is holomorphic;
- $\omega = \{(U_\alpha, f_\alpha)\}_\alpha$ on Y, where f_α is merom. satisfying $f_\beta = \det(J^{-1}\Psi_{\beta\alpha}) f_\alpha$ on $U_\alpha \cap U_\beta$

We consider the collection $\{(f_\alpha \circ \pi) \cdot \det(J\pi_\alpha)\}_\alpha$ on V_α and observe that

$$
\begin{aligned}
(f_\beta \circ \pi)\det(J\pi_\beta) &= (f_\alpha \circ \pi)\det(J^{-1}\Psi_{\beta\alpha})\det(J\pi_\beta) \\
&= (f_\alpha \circ \pi)\det J(\Psi_{\alpha\beta})\det J\pi_\beta \cdot \frac{\det J\varphi_{\beta\alpha}}{\det J\varphi_{\beta\alpha}} \\
&= (f_\alpha \circ \pi)\det(J\pi_\alpha)(\det J^{-1}\varphi_{\beta\alpha})
\end{aligned}
$$

$\Rightarrow$ the above collection of merom. functions define cocycles $\{\det J^{-1}\varphi_{\beta\alpha}\}$ $\Rightarrow$ it is a global meromorphic section of K_X:

$$K_X = \mathrm{div}(\{(f_\alpha \circ \pi)\det(J\pi_\alpha)\}_\alpha) =$$

$$= \operatorname{div}\big(\{\rho_\alpha \circ \pi\}_\alpha\big) + \operatorname{div}\big(\{\det(J\pi_\alpha)\}_\alpha\big)$$

$\overset{\shortparallel}{\pi^* K_Y}$ by def of pullback of a divisor

$$= \pi^* K_Y + \operatorname{div}\big(\{\det(J\pi_\alpha)\}_\alpha\big)$$

Let us consider a prime divisor V of X and its image W by π, let e its ramificat. index. By Local Normal Form theorem it there exist a local chart $\underline{x} = (x_1, \dots x_n)$ around a point of V such that $V = (x_1 = 0)$ and

$$\pi : \text{open set of } \mathbb{C}^n \longrightarrow \text{open set of } \mathbb{C}^n$$
$$(x_1, \dots x_n) \longmapsto (x_1^e, x_2, \dots x_n)$$

$$\Rightarrow \det J\pi_{\underline{x}} = \det \begin{pmatrix} e x_1^{e-1} & 0 & \cdots & 0 \\ & 1 & & 0 \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} = e x_1^{e-1}$$

$e-1$ is the coefficient of V in the divisor R. $\blacksquare$

§4.4 Maps to projective spaces

Def Let D be a divisor, we denote by $|D|$ the set of divisors of X lin. equivalent to D:

$$|D| := \left\{ D + \operatorname{div}(f) \;\middle|\; \begin{array}{l} D + \operatorname{div}(f) \geq 0 \\ f \text{ global merom. on } X \end{array} \right\}$$

When X is compact, then $|D|$ corresponds to the projective space of $H^0(X, \mathcal{O}_X(D))$.

A linear system P of X is a subspace of $|D|$, namely the set associated to a vector subspace W of $H^0(X, \mathcal{O}_X(D))$. We say that P is complete if $P = |D|$.

The dim. of P is $\dim P := \dim W - 1$.

The identification of P as $\mathbb{P}(H^0(X, \mathcal{O}_X(D))$ can be constructed as follows. Let s_0 be a global merom. section of $[D]$. Then

$$\begin{array}{ccc} |D| & \xrightarrow{\cdot s_0} & \mathbb{P}(H^0(X, \mathcal{O}_X(D))) \\ D + \mathrm{div}(f) & \longmapsto & f s_0 \end{array}$$

is a bijection (when X is compact)

Def We say that C is a fixed component of P if it is contained in any divisor D of P.

The fixed locus F of P is the biggest divisor contained in any divisor of P.

Clearly, for any $D \in P$, then $|D - F|$ has NO fixed components.

Def A base point $x \in X$ of P is a point contained in every divisor of P.

The base locus of P, denoted as $Bs(P)$, is the schematic set of base points of P

The codimension 1-part of $Bs(P)$ is the fixed component F of P.

Thm We have the following correspondence:

$$\left\{ \begin{array}{l} \text{rational maps } \phi : X \dashrightarrow \mathbb{P}^n \text{ with } \phi(X) \text{ NOT} \\ \text{contained in a hyperplane} \end{array} \right\} \Big/ \text{proj. transf.}$$

$$\updownarrow$$

$$\left\{ \begin{array}{l} \text{linear systems } P \text{ of } X \text{ without fixed part} \\ \text{of dimension } n \end{array} \right\}$$

proof Let P be a linear system without fixed part, thus P is a projective subspace of $\mathbb{P}(H^0(X, \mathcal{O}_X(D_0)))$ for some div. D_0. Let W be the associated vect. subsp. of P in $H^0(X, \mathcal{O}_X(D_0))$.

To any $x \in X$ we can associate the projective subspace of P of divisors passing on x:

$\{D \in P : x \in D\}$ is the proiettivizzation of $H_x := \operatorname{Ker}(f_x) \subseteq W \subseteq H^0(X, \mathcal{O}_X(D_0))$

where
$$f_x : W \longrightarrow \mathbb{C}$$
$$s \longmapsto s_\alpha(x) \quad \text{for } x \in U_\alpha$$

(Notice that f_x depends by U_α but $\ker(f_\alpha)$ does not!)

Clearly if x is not a base point, then f_x is surjective, so $\ker(f_x)$ is an hyperplane!

We have a (rational) map

$$\begin{aligned} X &\dashrightarrow \{\text{hyperplanes of } W\} \\ x &\longmapsto H_x \end{aligned}$$

However, we well-know that the set of hyperpl. of a vect. space V is exactly $\mathbb{P}(V^\vee)$:

$$\begin{aligned} \{\text{hyperpl. of } V\} &\longrightarrow \mathbb{P}(V^\vee) \\ H = \ker(f) &\longmapsto [f] \end{aligned}$$

Thus, we have a natural map:

$$\begin{aligned} X &\dashrightarrow \mathbb{P}(W^\vee) \\ x &\longmapsto [f_x] \end{aligned} \qquad (*)$$

whose indeterminacy locus is the base locus of P.

We can write this map in coordinates: given a basis $s_0, \dots s_n$ of W, then $(*)$ is

$$\begin{aligned} \phi_P : X &\dashrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \dots : s_n(x)] \end{aligned}$$

Notice that $\phi_p(X)$ is NOT contained in an hyperplane, otherwise if

$$\phi_p(X) \subseteq \{\underline{z} : \textstyle\sum a_i z_i = 0\}$$

then $\sum_i a_i s_i(x) = 0 \quad \forall x \in X \setminus Bs(x)$

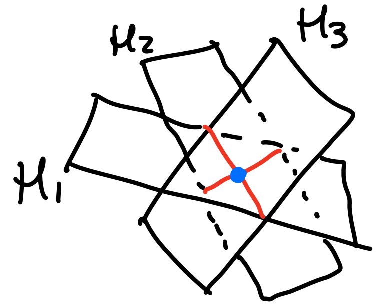
$\Rightarrow \forall x \in X$

$\Rightarrow \sum a_i s_i = 0 \Rightarrow s_0, \dots, s_n$ is NOT a basis ↯

Clearly, if we change ϕ_p using another basis of W then we obtain just a map that is a composition of ϕ_p with a proj. transf. of $\mathbb{P}^n$.

Conversely, consider a rational map

$\phi : X \dashrightarrow \mathbb{P}^n$ that is not contained in a hyperplane, namely $\phi^* H$ is well-defined.
We notice that $\phi^* H$ has not fixed part because we can move hyperplane sections on $\mathbb{P}^n$ (and so their pullbacks on X) such that two of them intersect in a codim. 2 subspace.
Finally, if $\underset{(x_i = 0)}{H_i} = \{(U_{x_j}, \frac{x_i}{x_j})\}_j$, then



$\phi^* H_i = \{(\phi^{-1}(U_{x_j}), (\frac{x_i}{x_j}) \circ \phi)\}_j$ and they are lin. indep. global holomorphic sections of $|\phi^* H|$ $\Rightarrow$

$$\phi_{\phi^* H} : X \dashrightarrow \mathbb{P}^n$$
$$x \longmapsto \left[\frac{x_1}{x_j}\circ\phi : \dots \underset{\underset{\phi(x)}{\|}}{1} : \dots \frac{x_n}{x_j}\circ\phi\right] \text{ on } U_{x_j}$$

$\boxtimes$

Def Given a smooth proj. variety X, the map induced by $|K_X|$ is called CANONICAL MAP, when $p_g(X)\geq 2$:

$$\phi_{K_X} : X \dashrightarrow \mathbb{P}^{p_g(X)-1}$$

- A pluricanonical divisor is a divisor lin. equivalent to a multiple of a canonical divisor.

The pluricanonical class of X is nK_X in $\mathrm{Pic}(X)$, $n\in\mathbb{N}$, where K_X is the can. class.

The plurigenus of X is $P_n := h^0(X, nK_X)$

The pluricanonical map of X is the map induced by $|nK_X|$:

$$\phi_{nK_X} : X \dashrightarrow \mathbb{P}^{P_n - 1}$$

The structure of canonical and pluricanonical maps are studied a lot in the literature from seminal works of Enriques, Kodaira, Bombieri, ecc...

§4.5 Iitaka dimension of a Divisor

Def Let D be a divisor of X (smooth proj. variety). If

$$h^0(X, mD) = 0 \quad \forall m \in \mathbb{Z}_{>0}$$

then we say that the Itaka dim. of D is $\kappa(X,D) = -\infty$.

Otherwise, we say that

$$\kappa(X,D) := \max_m \dim\left(\Phi_{mD}(X)\right)$$

Rem $\kappa(X,D) \in \{-\infty, 1, \dots \dim(X)\}$.

Def D (or equivalently its associated line bundle $[D]$) is called big $\overset{def}{\iff}$ $\kappa(X, D) = \dim(X)$.

Thm The Iitaka dimension $k = \kappa(X,D) \neq -\infty$ is the smallest number for which

$$\limsup_{m \to +\infty} \frac{h^0(X, mD)}{m^k} < +\infty$$

Def $\kappa(X) := \kappa(X, K_X)$ is called Kodaira dimension of X.

FUN FACT: $\kappa(X,D)$ is a birational invariant!

IMPORTANT In general, for a smooth proj. variety X

$q(X) = h^1(X, \mathcal{O}_X)$, $p_g(X) = h^0(X, K_X)$

irregularity geom. genus

$P_n := h^0(X, nK_X)$, $n \geq 2$ $\kappa(X)$

n-th plurigenus Kodaira dim.

are birational invariants.